Find $\mathscr{L}\{\, u(t-1)\, t^3 + u(t-\frac{\pi}{2})\cos t \,\}$. **SCORE: _____ / 15 PTS**

$$= e^{-s}\mathcal{L}\{\underbrace{(t+1)^3}_{2}\} + e^{-\frac{\pi}{2}s}\mathcal{L}\{\underbrace{\cos(t+\tfrac{\pi}{2})}_{2}\}$$

$$= e^{-s}\mathcal{L}\{\underbrace{t^3+3t^2+3t+1}_{1}\} + e^{-\frac{\pi}{2}s}\mathcal{L}\{\underbrace{-\sin t}_{1}\}$$

$$= \underbrace{e^{-s}}_{2}\left(\underbrace{\frac{6}{s^4}+\frac{6}{s^3}+\frac{3}{s^2}+\frac{1}{s}}_{4}\right) - \underbrace{e^{-\frac{\pi}{2}s}}_{2}\left(\underbrace{\frac{1}{s^2+1}}_{1}\right)$$

Find $\mathscr{L}^{-1}\left\{\dfrac{28s+32}{(s^2+4s)(s^2+4)}\right\}$.

SCORE: _____ / 35 PTS

$$\frac{28s+32}{s(s+4)(s^2+4)} = \frac{A}{s} + \frac{B}{s+4} + \frac{Cs+D(2)}{s^2+4}$$

$$28s+32 = A(s+4)(s^2+4) + Bs(s^2+4) + Cs^2(s+4) + 2Ds(s+4)$$

$s=0$: $32 = 16A \longrightarrow A=2$

$s=-4$: $-80 = -80B \longrightarrow B=1$

COEF OF $s^3$: $0 = A+B+C \longrightarrow C=-A-B=-3$

COEF OF $s^2$: $0 = 4A+4C+2D \longrightarrow D=-2A-2C=2$

$$\mathscr{L}^{-1}\left\{\underbrace{\frac{2}{s}} + \underbrace{\frac{1}{s+4}} + \underbrace{\frac{-3s}{s^2+4}} + \underbrace{\frac{2(2)}{s^2+4}}\right\} \quad 4 \text{ EACH}$$

$$= \underbrace{2}_{4} + \underbrace{e^{-4t}}_{4} \underbrace{-3\cos 2t}_{4} + \underbrace{2\sin 2t}_{7}$$

Use Laplace transforms to solve the IVP $y'' + 2y' + y = 12te^{-t}, \quad y(0) = -4, \quad y'(0) = 3$. **SCORE: _____ / 40 PTS**

$$s^2 \mathcal{L}\{y\} - sy(0) - y'(0) + 2s\mathcal{L}\{y\} - 2y(0) + \mathcal{L}\{y\} = \frac{12}{(s+1)^2}$$

$$\underbrace{(s^2+2s+1)\mathcal{L}\{y\}}_{4} + \underbrace{4s+5}_{4} = \underbrace{\frac{12}{(s+1)^2}}_{}\ 4$$

$$\mathcal{L}\{y\} = \underbrace{\frac{-4s-5}{(s+1)^2} + \frac{12}{(s+1)^4}}_{}\ 4$$

$$= \frac{-4(s+1)-1}{(s+1)^2} + \frac{12}{(s+1)^4}$$

$$= \underbrace{-\frac{4}{s+1}}_{4} \underbrace{- \frac{1}{(s+1)^2}}_{4} + \frac{12}{(s+1)^4}$$

$$y = \underbrace{-4e^{-t}}_{4} - \underbrace{te^{-t}}_{4} + \underbrace{2t^3e^{-t}}_{8}$$

Find two linearly independent series solutions of $4x^2y'' + x(x+6)y' - 2y = 0$ about the point $x = 0$. **SCORE: _____ / 60 PTS**

**<u>You must find the recurrence relation for the coefficients</u>**
**<u>and you must give the first three non-zero terms of each series (or all terms if a solution has fewer than three non-zero terms),</u>**
**<u>but you do NOT need to write your final answers in sigma notation.</u>**

(2) $y'' + \frac{x+6}{4x}y' - \frac{1}{2x^2}y = 0$ $\quad x=0 \longrightarrow$ SINGULAR POINT $\downarrow$ REGULAR SINGULAR POINT

$$\lim_{x\to 0} \frac{x(x+6)}{4x} = \frac{3}{2} \qquad \lim_{x\to 0} \frac{-x^2}{2x^2} = -\frac{1}{2}$$

$$\underbrace{r(r-1)+\tfrac{3}{2}r}_{(2)}\underbrace{-\tfrac{1}{2}}_{(2)}=0 \longrightarrow \underbrace{r^2+\tfrac{1}{2}r-\tfrac{1}{2}=0}_{(2)} \longrightarrow (r+1)(r-\tfrac{1}{2})=0$$

$$r=-1, \tfrac{1}{2} \quad (4)$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r} \qquad y'=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$4x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \; (3) + (x^2+6x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \; (3) - 2\sum_{n=0}^{\infty}a_n x^{n+r} \; (3)$$

$$= \sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^{n+r} \; (3) + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} \; (3) + \sum_{n=0}^{\infty} 6(n+r)a_n x^{n+r} \; (3) - \sum_{n=0}^{\infty} 2a_n x^{n+r} \; (3)$$

$$= \sum_{n=0}^{\infty}[4(n+r)(n+r-1)+6(n+r)-2]a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} \; (4)$$

$$= \underbrace{(4r(r-1)+6r-2)}_{4r^2+2r-2=0}a_0x^r \; (4) + \sum_{n=1}^{\infty}\left[4(n+r)(n+r-1)+6(n+r)-2]a_n + (n+r-1)a_{n-1}\right]x^{n+r} = 0 \; (3)$$

$r^2+\frac{1}{2}r-\frac{1}{2}=0$

SAME AS ABOVE

(4) $$a_n = \frac{-(n+r-1)}{4(n+r)(n+r-1)+6(n+r)-2}a_{n-1}, \quad n \geq 1$$

$r=-1$: $a_n = \dfrac{(2-n)a_{n-1}}{4(n-1)(n-2)+6(n-1)-2}$

$a_1 = \frac{1}{-2}a_0$ (2)

(2) $a_2 = 0 = a_3 = a_4 = \dots$

$y_1 = a_0x^{-1}(1-\frac{1}{2}x)$

$= a_0(x^{-1}-\frac{1}{2})$ (2)

$r=\frac{1}{2}$: $a_n = \dfrac{(\frac{1}{2}-n)a_{n-1}}{4(n+\frac{1}{2})(n-\frac{1}{2})+6(n+\frac{1}{2})-2}$

$= \dfrac{(\frac{1}{2}-n)a_{n-1}}{4n^2+6n}$

$a_1 = \dfrac{-\frac{1}{2}}{10}a_0 = -\frac{1}{20}a_0$ (2)

$a_2 = \dfrac{-\frac{3}{2}}{28}a_1 = -\frac{3}{1120}a_0$ (2)

$y_2 = a_1\sqrt{x}(1-\frac{1}{20}x-\frac{3}{1120}x^2 \dots)$ (2)